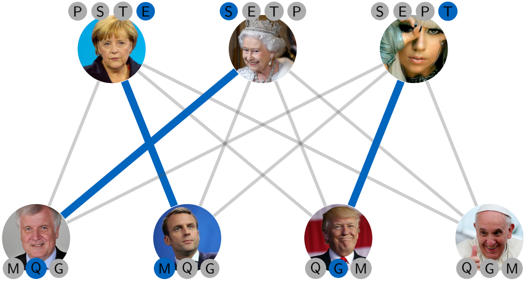
# Stable Matchings

Preferences of market participants are expressed by total orders (preference lists) instead of numerically via costs.

# Matching with Preferences




Preference list: For each node $v \in V$, a total order $\prec_v$ of the neighbors $N_G(v)$ is given.

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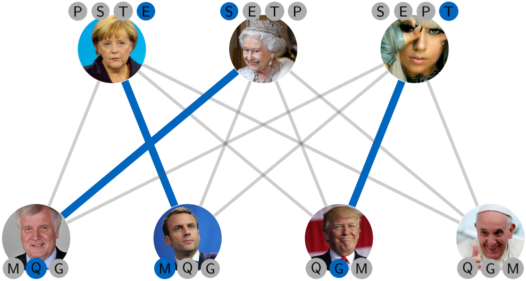


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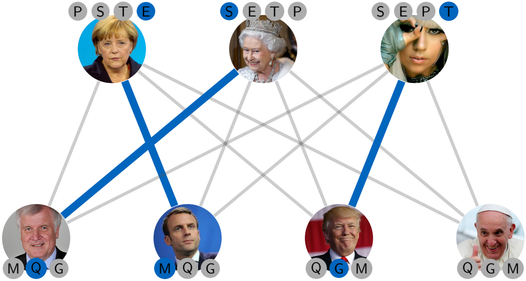


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Represent matching $M$ as a mapping $\mu : V \to V \cup \emptyset$ with

$$\mu(v) := \begin{cases} \emptyset & \text{if } v \text{ is not covered by } M, \\ u & \text{if } \{u, v\} \in M. \end{cases}$$

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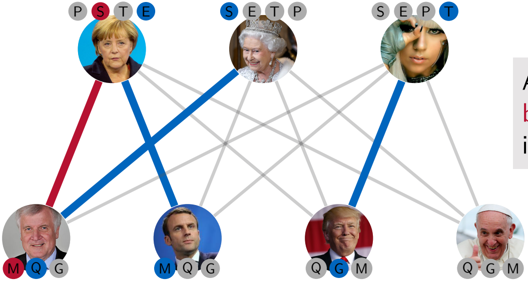


An edge $\{u, v\} \in E \setminus M$ is called blocking (w.r.t. $M$), if $u \prec_v \mu(v)$ und $v \prec_u \mu(u)$.

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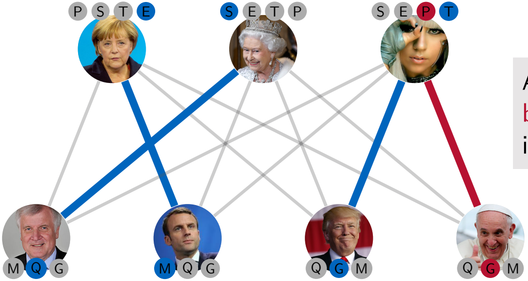


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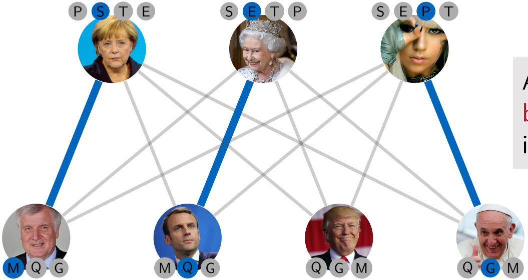


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**Definition.** A matching $M$ is stable, if there is no blocking edge w.r.t. $M$.

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- Input: Graph $G = (V, E)$; every $v \in V$ has a preference list over their neighbors $N_G(v)$.
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Depends for whom...

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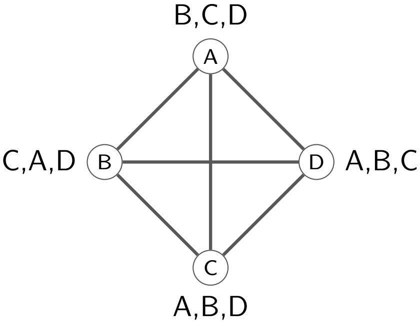


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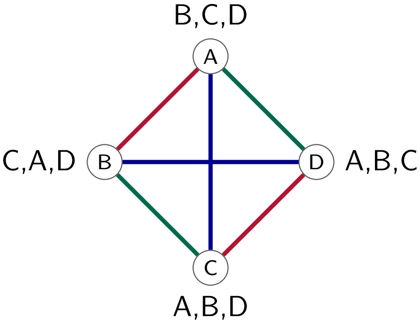


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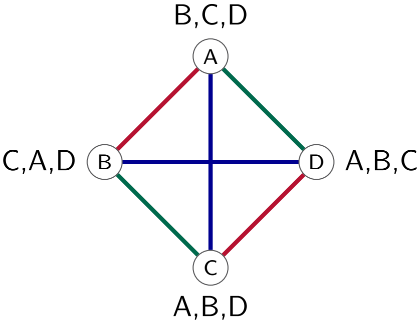


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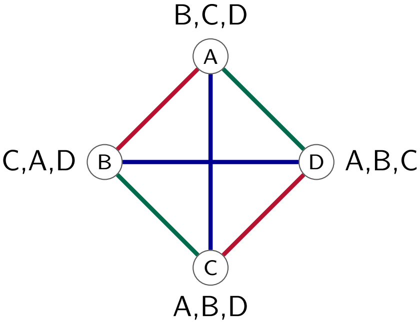


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red: B,C

blue: A,B

green: C,A

# Stable Matchings in (complete) bipartite graphs

stable marriage problem

# Approach 1: Enumeration

- Algorithm: Consider all possible (maximum) matchings and check every matching for stability.
  - How to find? How to check?
- Advantage: The algorithm is correct.
- Disadvantage: There are too many perfect matching ($n!$), hence the algorithm is not efficient.

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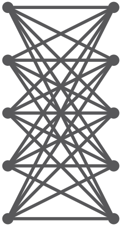
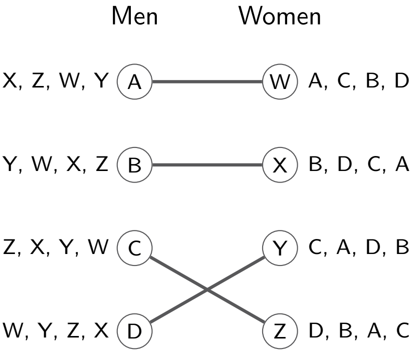


**Example**

A complete bipartite graph with 5 vertices on each side has $5! = 5 * 4 * 3 * 2 * 1 = 120$ distinct perfect matchings.

# Approach 2: Greedy

**Greedy**: Given a matching $M$. As long as there is a blocking edge $\{m, w\} \notin M$, exchange the matching edges $\{m, w'\}$ and $\{m', w\}$ by $\{m, w\}$ and $\{m', w'\}$.

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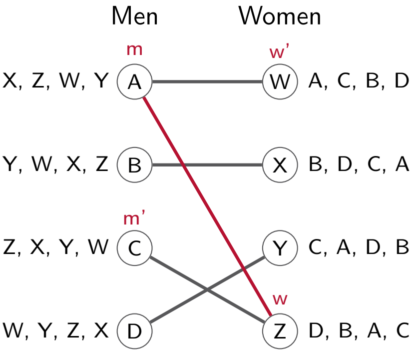


{A,Z} unhappy

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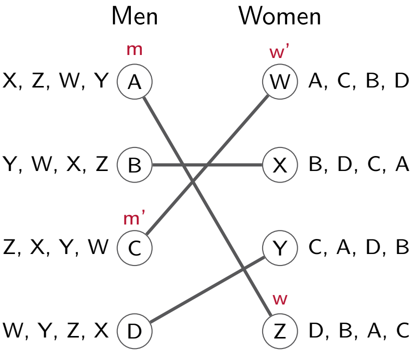

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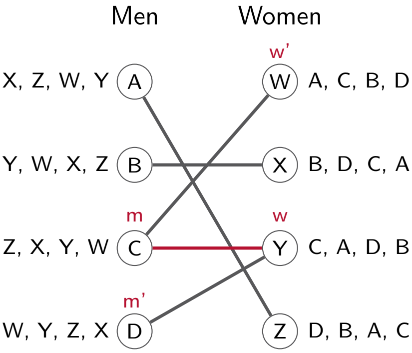


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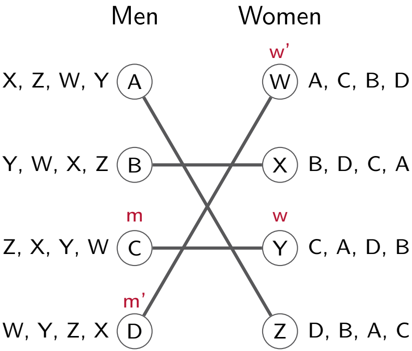

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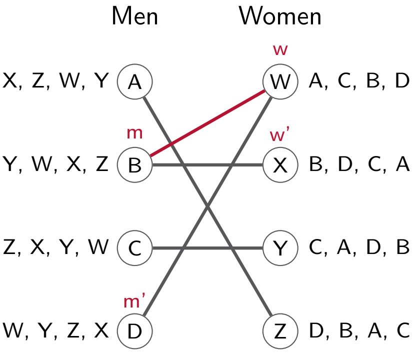


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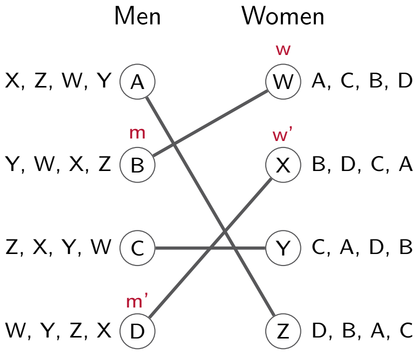

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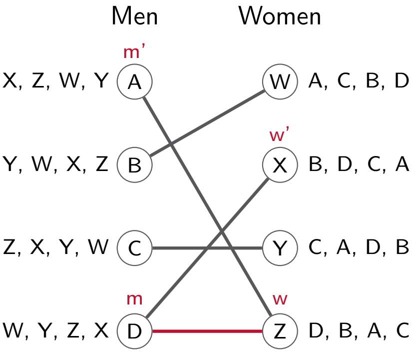


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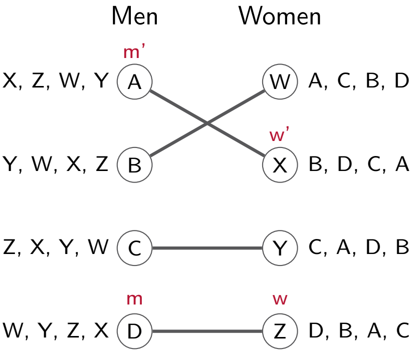

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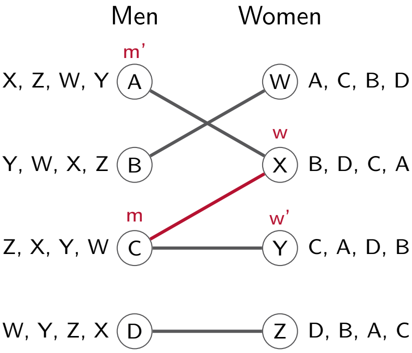


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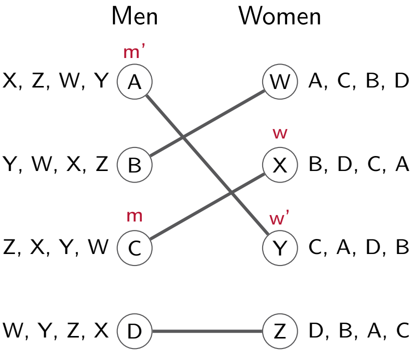

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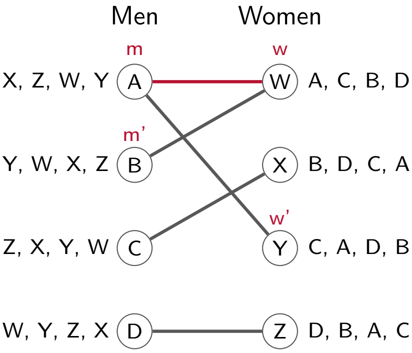


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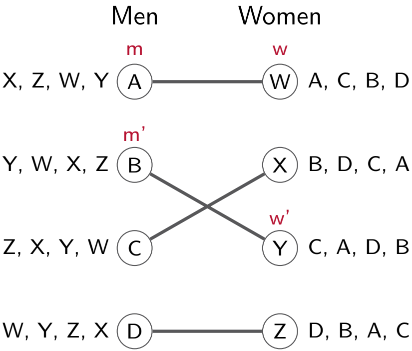

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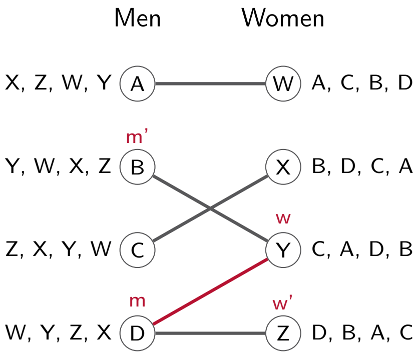


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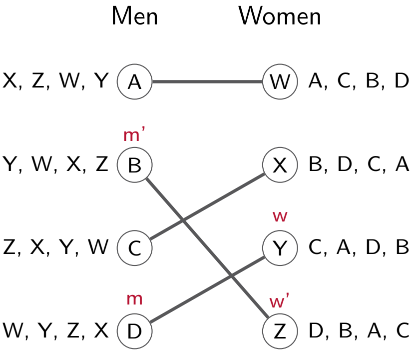

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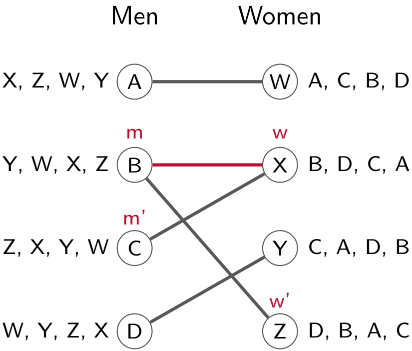


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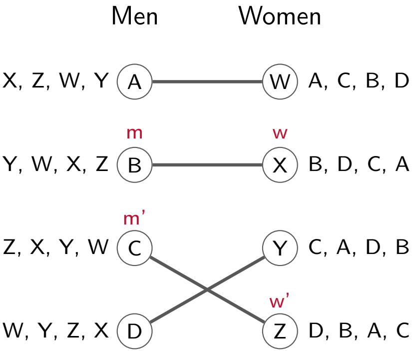

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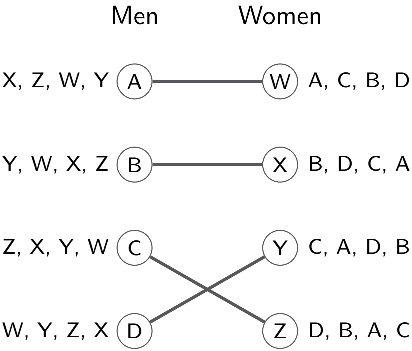


Alert: We are again at the beginning! Infinite loop!

# Gale-Shapley Algorithm (1962)

(aka Deferred-Acceptance Algorithm)

Originally for the more general College Admission Problem.

# 3rd Approach: Gale-Shapley Algorithm

**Input** : Bipartite graph $G = (A \dot\cup B, E)$, preference lists $\prec_v$, $v \in V = A \cup B$.
**Output** : A stable matching in $G$.

Initialization: All men $A$ and women $B$ are unengaged $\mu(v) = \emptyset$, for all $v \in V$.
**while** There exists a man $m \in A$ who is currently unengaged $(\mu(m) = \emptyset)$ and has not yet proposed to all women **do**
  Choose any such man $m$.
  $m$ proposes to the best woman $w = \min_{\prec_m}\{w' \in B\}$ on his list.
  **if** woman $w$ is not yet engaged **then**
    Engage $m$ and $w$, i.e., set $\mu(m) = w, \mu(w) = m$.
  **else**
    **if** $w$ prefers $m$ over her current fiancé $m'$, $m \prec_w \mu(w)$ **then**
      Break the engagement between $w$ and $m'$ $(\mu(w) = \mu(m') = \emptyset)$.
      Engage $m$ and $w$ $(\mu(m) := w, \mu(w) = m)$.
      Remove $w$ from $m'$'s list.
    **else**
      Remove $w$ from $m$'s list.

**return** Matching $M^*$ consisting of all currently engaged pairs.

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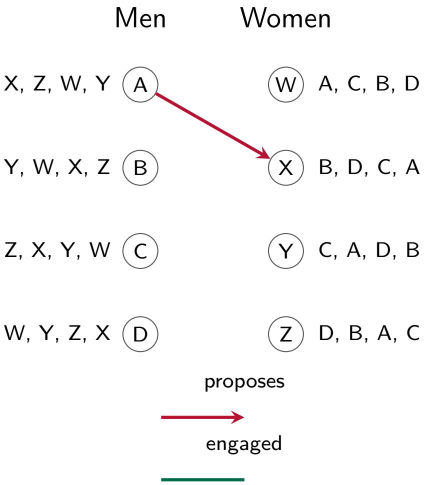
# Gale-Shapley Algorithm: Example 1




- $A$ proposes to $X$

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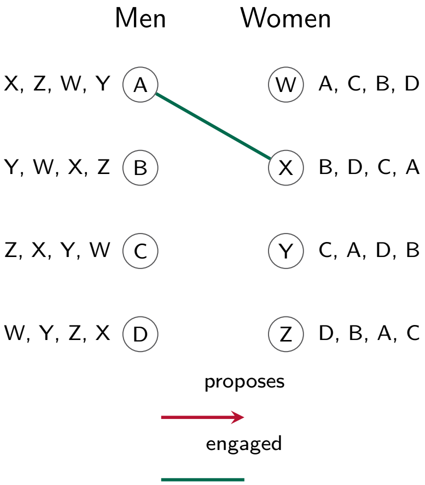


- $A$ proposes to $X$
- $X$ accepts and gets engaged to $A$

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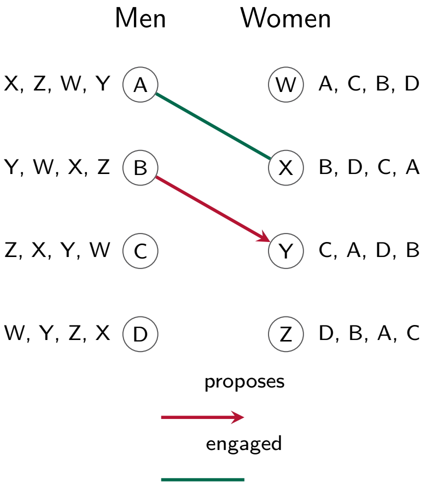


- $B$ proposes to $Y$

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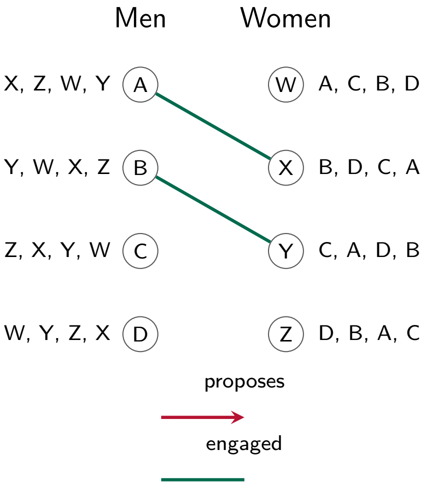


- $B$ proposes to $Y$
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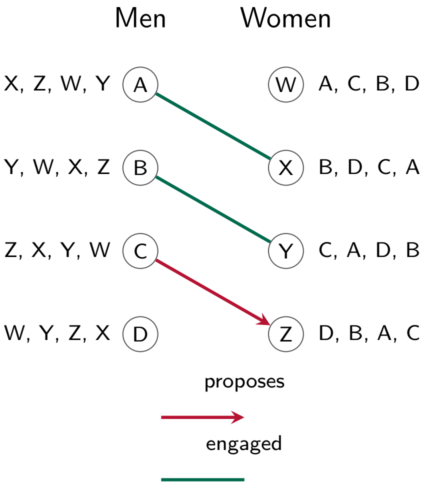


- $C$ proposes to $Z$

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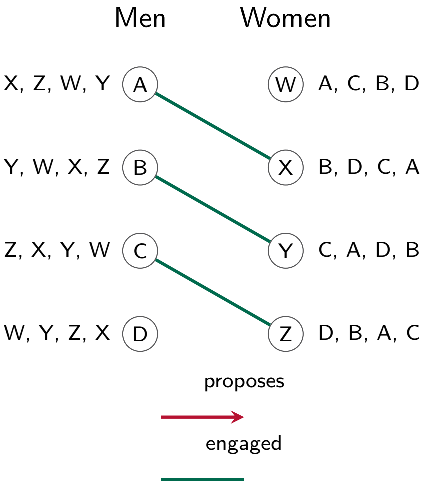


- $C$ proposes to $Z$
- $Z$ accepts and gets engaged to $C$

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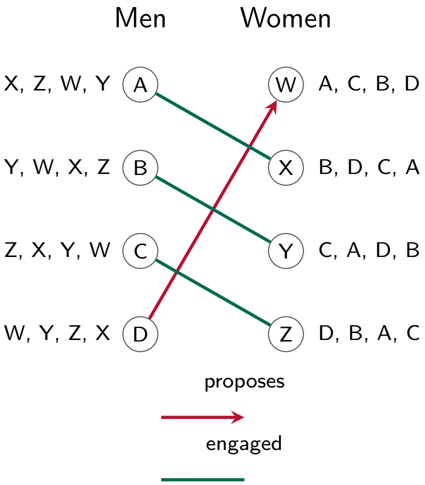


- *D* proposes to *W*

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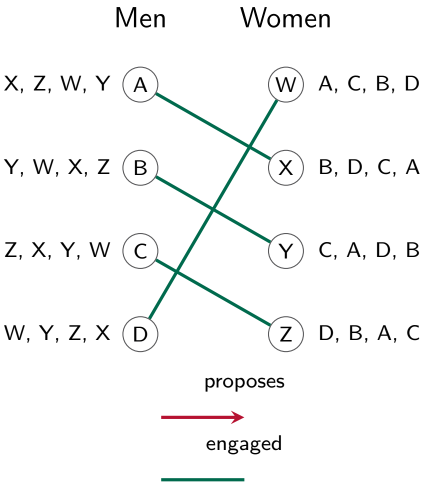


- *D* proposes to *W*
- *W* accepts and gets engaged to *D*

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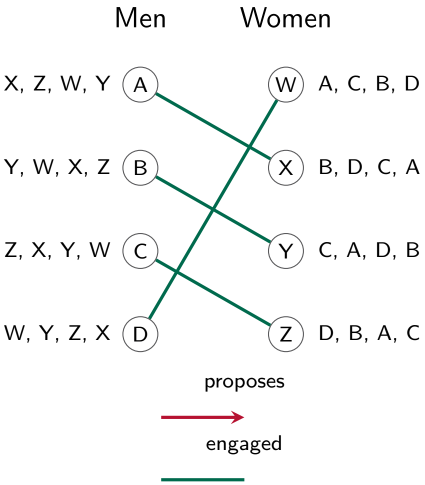


- Alle men are engaged.

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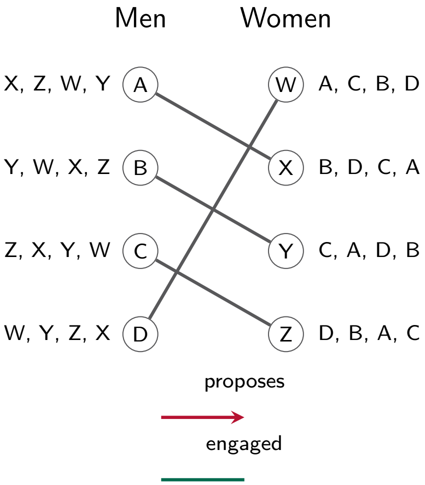


- Alle men are engaged.
- The resulting matching is perfect and stable.

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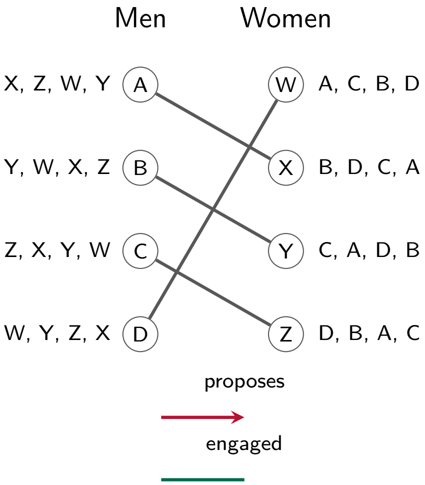
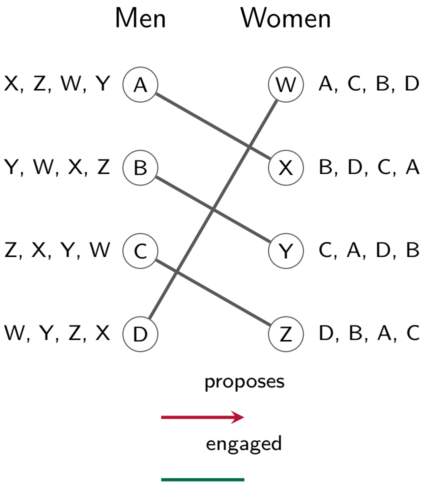



- Alle men are engaged.
- The resulting matching is perfect and stable.
- Every man has his 1. priority; not the women.

# Gale-Shapley Algorithm: Example 1

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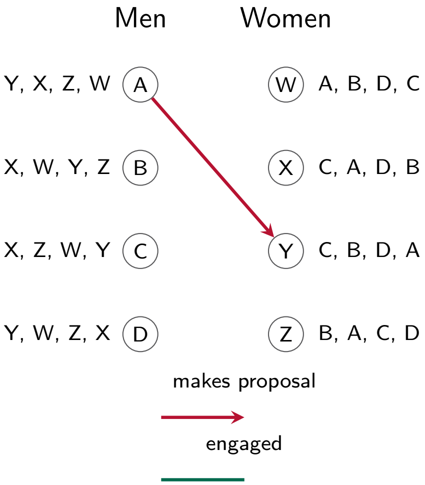
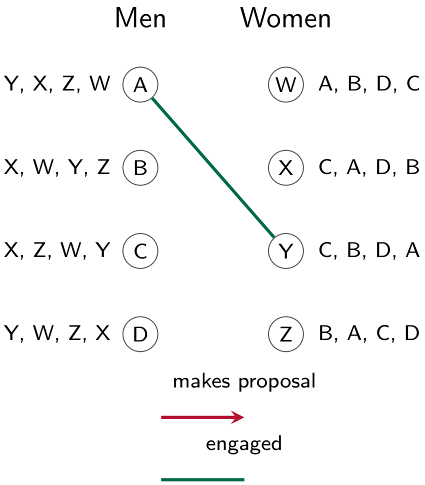
Boring Example (one round).

# Gale-Shapley Algorithm: 2nd Example



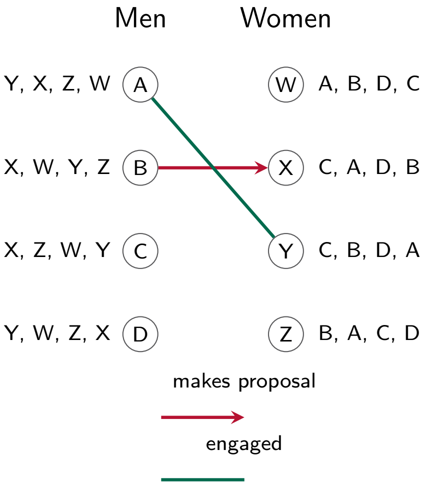

- $A$ proposes to $Y$

# Gale-Shapley Algorithm: 2nd Example




- *A* proposes to *Y*
- *Y* accepts and gets engaged to *A*

# Gale-Shapley Algorithm: 2nd Example




- $B$ proposes to $X$

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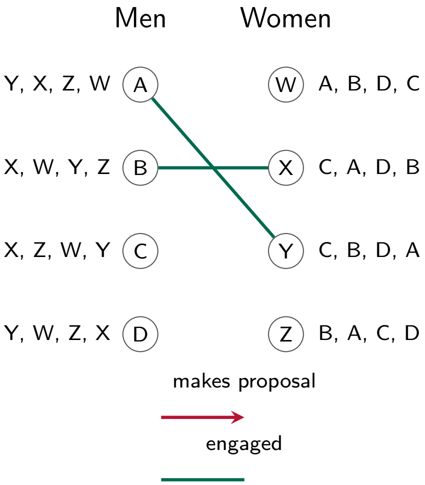


- *B* proposes to *X*
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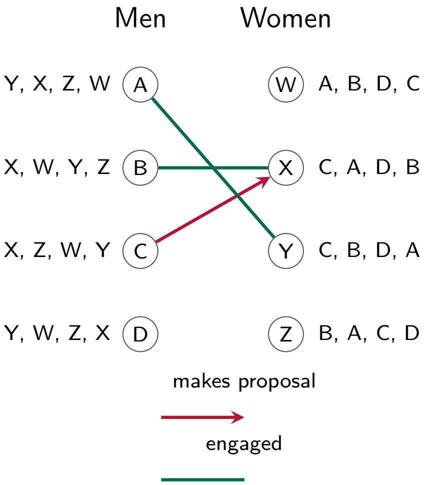
# Gale-Shapley Algorithm: 2nd Example




- $C$ proposes to $X$

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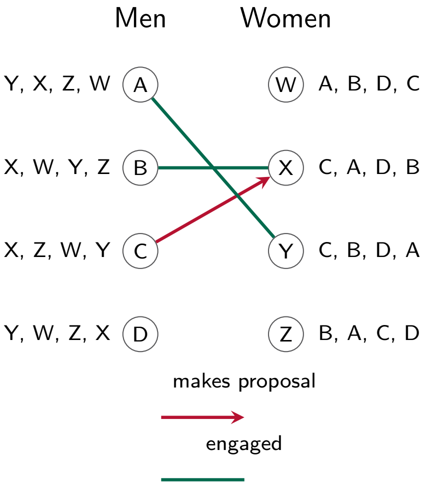
# Gale-Shapley Algorithm: 2nd Example




- $C$ proposes to $X$
- $X$ compares $C$ and $B$: she prefers $C$ over $B$

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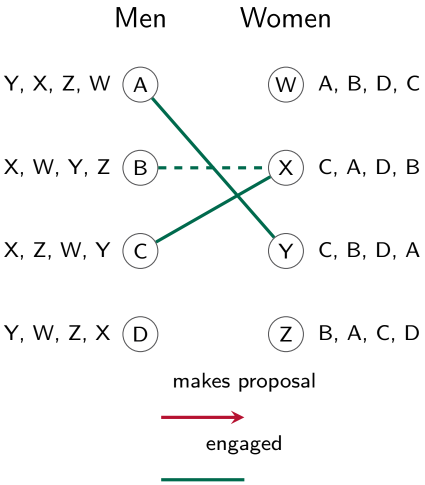
# Gale-Shapley Algorithm: 2nd Example




- *C* proposes to *X*
- *X* compares *C* and *B*: she prefers *C* over *B*
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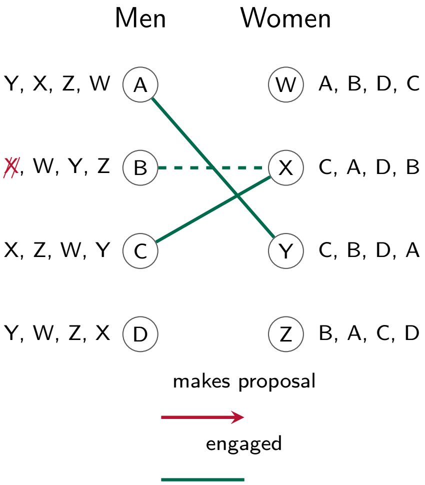
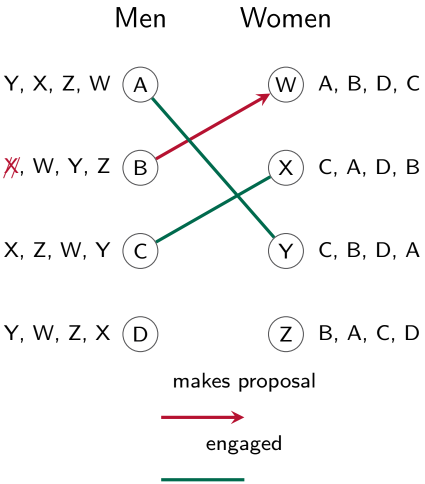



- *C* proposes to *X*
- *X* compares *C* and *B*: she prefers *C* over *B*
- *X* breaks engagement with *B* and gets engaged to *C*
- *B* removes *X* from his list

# Gale-Shapley Algorithm: 2nd Example




- *B* proposes to *W*

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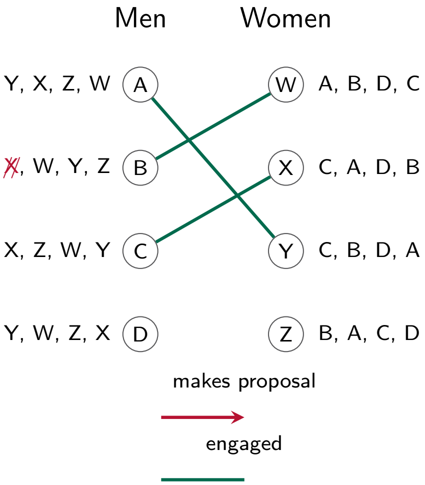
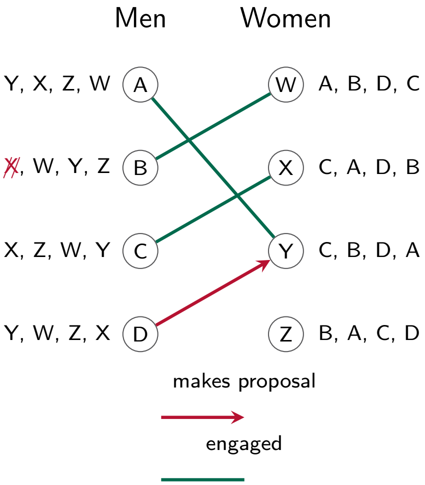


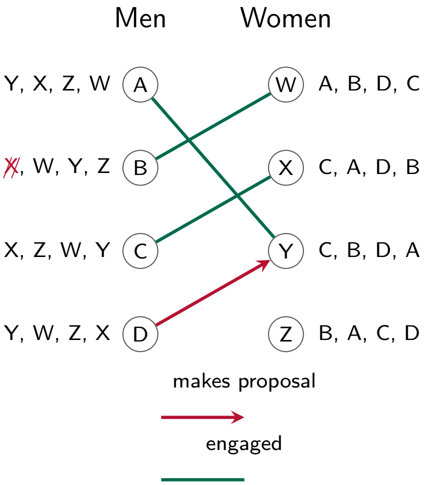

- *B* proposes to *W*
- *W* accepts and gets engaged to *B*

# Gale-Shapley Algorithm: 2nd Example




- *D* proposes to *Y*

# Gale-Shapley Algorithm: 2nd Example




- *D* proposes to *Y*
- *Y* compares *D* and *A*: she prefers *D* over *A*

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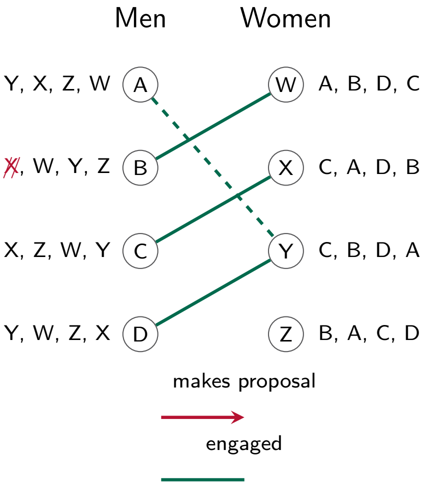


- *D* proposes to *Y*
- *Y* compares *D* and *A*: she prefers *D* over *A*
- *Y* breaks engagement with *A* and gets engaged to *D*

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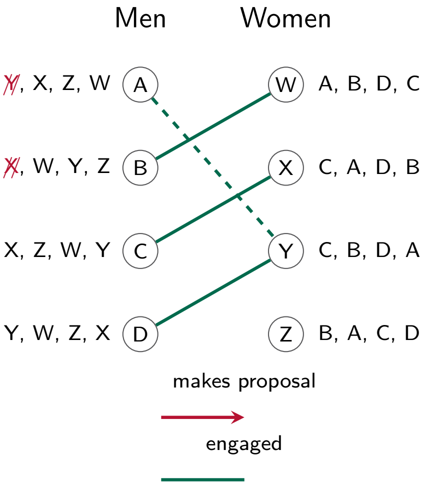
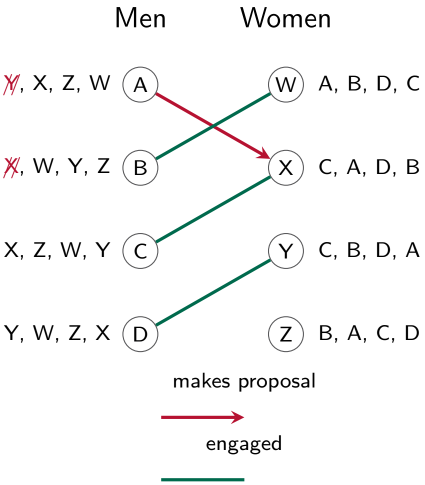
# Gale-Shapley Algorithm: 2nd Example




- *D* proposes to *Y*
- *Y* compares *D* and *A*: she prefers *D* over *A*
- *Y* breaks engagement with *A* and gets engaged to *D*
- *A* removes *Y* from his list

# Gale-Shapley Algorithm: 2nd Example




- $A$ proposes to $X$

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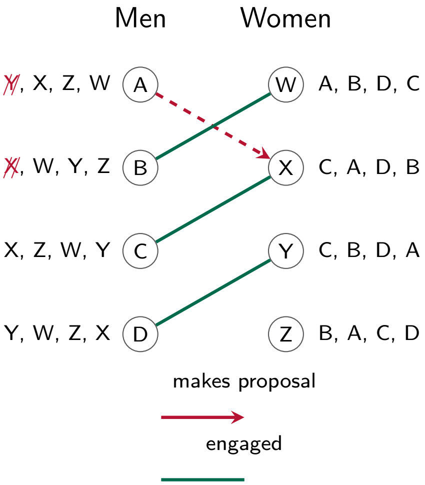
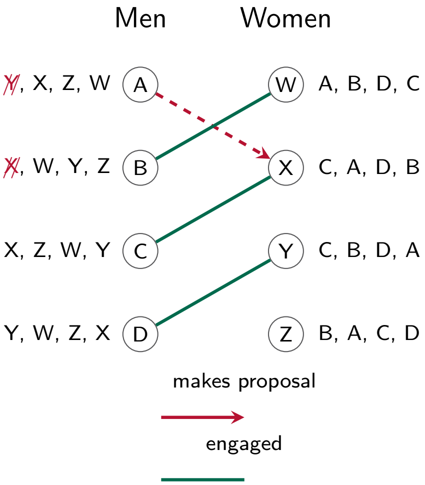



- *A* proposes to *X*
- *X* prefers *C* over *A*

# Gale-Shapley Algorithm: 2nd Example

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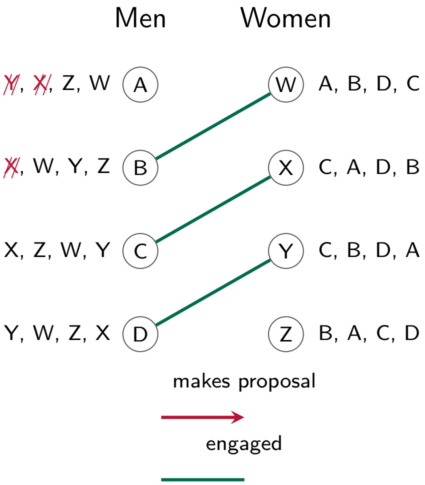


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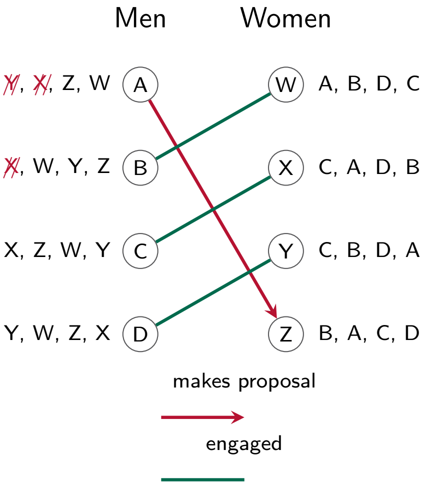


- $A$ proposes to $Z$

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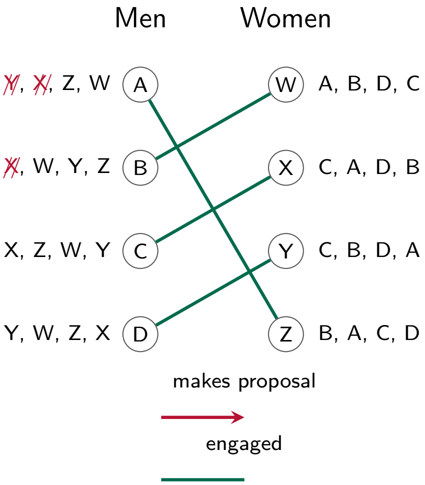
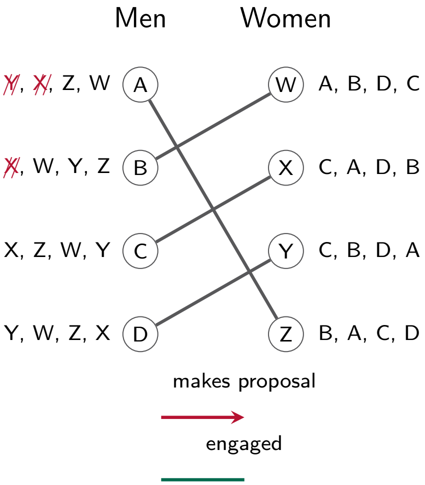



- *A* proposes to *Z*
- *Z* is overjoyed and accepts
- *Z* and *A* get engaged

# Gale-Shapley Algorithm: 2nd Example




- Termination condition reached: all men are engaged

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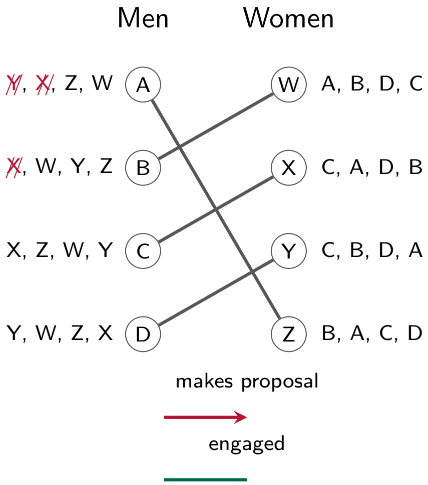
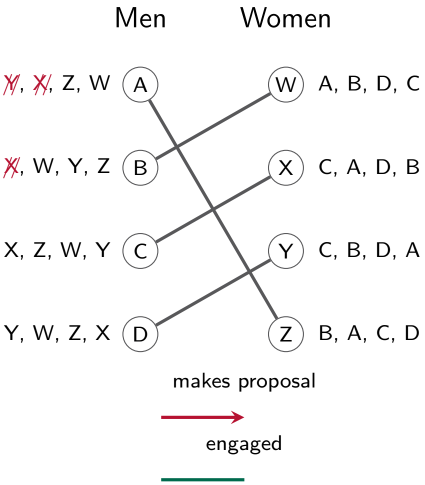



- Termination condition reached: all men are engaged
- The algorithm outputs the current matching

# Gale-Shapley Algorithm: 2nd Example




## Observation

- The resulting matching is perfect and stable
- Neither all men nor all women are matched with their first choice

# Correctness – Termination

**Lemma**

The algorithm terminates after $\mathcal{O}(|A||B|)$ steps.

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- Since there are only $|A||B|$ possible man-woman pairs, the statement follows. $\square$

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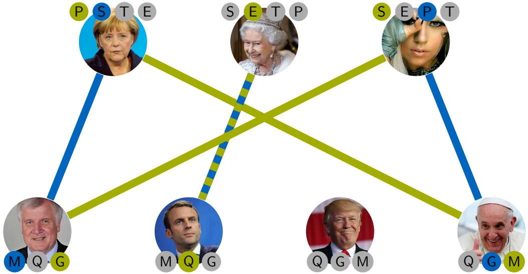
**Theorem**

Gale-Shapley's algorithm computes a stable matching in $\mathcal{O}(|A||B|)$.

# The Best Stable Matching?

Suppose there are multiple stable matchings in a graph — which one does the Gale-Shapley algorithm find?

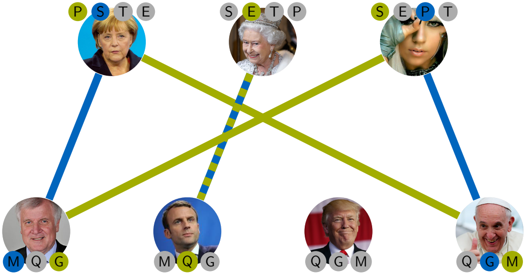
# Best stable Matching?




Blue Matching is better for  and .

Green Matching is better for  and .

# Best stable Matching?




**Definition.** We call a woman $w$ a valid partner of man $m$, if there is a stable matching $M$ with $(w, m) \in M$.

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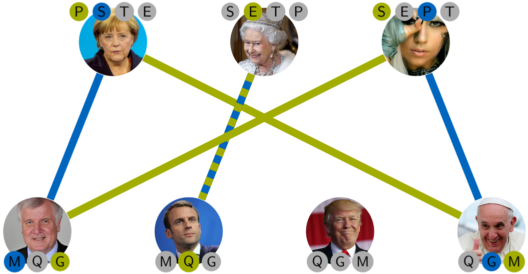


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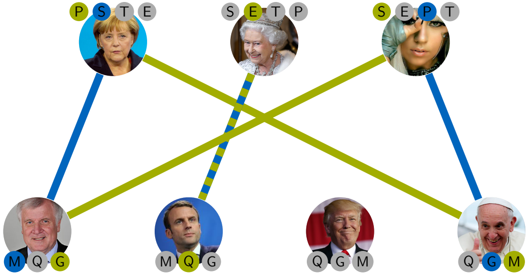


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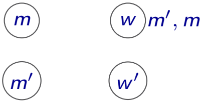

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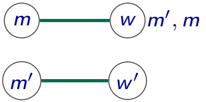


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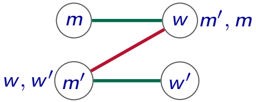


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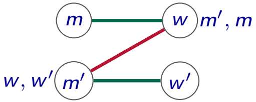
- Contradiction! $M$ is not a stable matching. □




stable Matching $M$
aber blocking pair $\{m', w\}$ in $M$

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**Satz**

The Gale-Shapley Algorithm matches to each woman her worst valid partner.

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- Suppose there is a woman $w$ who is engaged with a man $m'$, who is not the worst valid partner $m$ for her.

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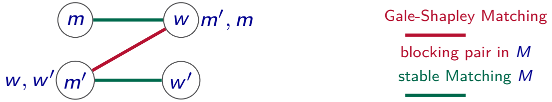

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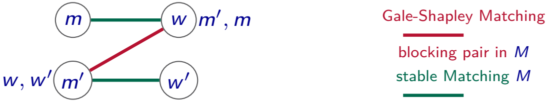
- Contradiction to the stability of $M$, since $\{m', w\}$ is a blocking pair. □



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**Frage:**

Can it be strategically worth to give false preferences?

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Suppose the participants know that

- the matching is computed by Gale-Shapley
- and know the preferences of the remaining participants.

| | | | |
|---|---|---|---|
| X, Y, Z | A | X | B, A, C |
| Y, X, Z | B | Y | A, B, C |
| X, Y, Z | C | Z | A, B, C |

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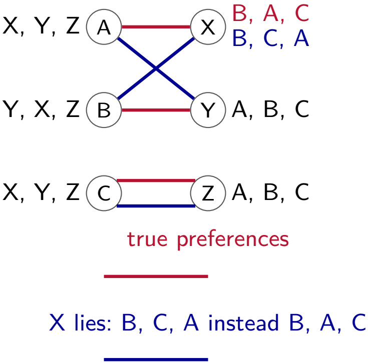
**Answer:**

„No“ for the men, „yes“ for the women.



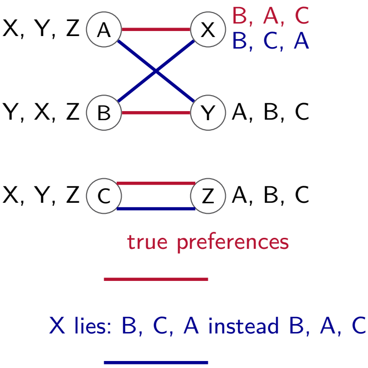
# Strategically manipul manipulable?

**Frage:**

Can it be strategically worth to give false preferences?

Suppose the participants know that

- the matching is computed by Gale-Shapley
- and know the preferences of the remaining participants.



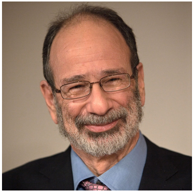

**Answer:**

„No“ for the men, „yes“ for the women.

The algorithm is “one-sided” strategy-proof.

# Nobel-price for Economics 2012

- Lloyd S. Shapley (1923-2016) together with Alvin E. Roth
- „. . . for the theory of stable allocations and the practice of market design."
- Shapley: theoretical foundations, Roth: Exchange-market for kidney exchange, assignment of students to High Schools



Alvin E. Roth, 2012

Lloyd Shapley, 1980

# Recap Matchings

- Weighted bipartite matching
  - Weighted directed graph to compute augmenting paths
- Non-bipartite matchings
  - Augmenting paths vs. blossoms
- Edmonds-Gallai Decomposition
  - Structure of maximum matchings
- Stable matchings
  - preference list for matching participants